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RESEARCH MEMORANDUM

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ON MULTI-STAGE GAMES WITH IMPRECISE PAYOFF

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RM-1337

9 September 1954

Assigned to

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

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Summary: It is shown that under certain natural conditions the play of a multi-stage game is to a great extent independent of the payoff function.

ON MULTI-STAGE GAMES WITH IMPRECISE PAYOFF

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§1. Introduction

There is a large class of situations of economic and military significance which can be considered to be multi-stage games. In many of these situations, the payoff function is easily determined; in others, it is a matter of difficulty to determine a suitable criterion.

The purpose of this note is to show, in a heuristic fashion, that in many cases the optimal play is independent of the precise form of the payoff, provided only that this payoff possesses certain intuitive properties.

§2. Description of the Multi-stage Game

Let us consider a zero-sum multi-stage game where each play is determined by the game matrix $A = (a_{ij})$. Let initially the first player possess a quantity x of resources and the second player a quantity y. Since the game is zero sum, the state of the game at any time is described by x.

Defining a suitable criterion function, let $f(x)$ represent the value of the game to the first player.

Then, f(x) satisfies the functional equation

$$
\begin{aligned}
f(x) &= \operatorname*{Min}_{q}\ \operatorname*{Max}_{p} \sum_{i,j} p_i q_j\, f(x + a_{ij}) \\
&= \operatorname*{Max}_{p}\ \operatorname*{Min}_{q} \sum_{i,j} p_i q_j\, f(x + a_{ij})
\end{aligned}
\tag{2.1}
$$

(see [1], [3]). The quantities p_i and q_j will, in general, depend upon x.

§3. Assumptions Concerning f(x)

Let us now assume that x and y are large compared to a_{ij}. In other words, the state of the system is only slightly disturbed by any one play of the game. Furthermore, let us assume that the value of the matrix A is zero, which is to say, it is on the average a fair game. Otherwise, the play is relatively trivial.

Finally, we assume that it pays to start with a larger initial resource. Then

$$f'(x) > 0 \tag{3.1}$$

§4. Heuristic Conclusion

Under these assumptions, we wish to show plausibly but not rigorously that the p_i and q_j are approximately independent of x and f(x). This means that, under these assumptions, on each

play the players attempt to maximize the single-stage return, the ordinary expected value.

Let us write

$$f(x + a_{ij}) \cong f(x) + a_{ij}f'(x) \tag{4.1}$$

Then, from (2.1),

$$f(x) \cong \underset{q}{\mathrm{Min}}\ \underset{p}{\mathrm{Max}} \sum_{i,j} p_i q_j \left[f(x) + a_{ij}f'(x) \right] \tag{4.2}$$

or

$$f(x) \cong f(x) \sum_{i,j} p_i q_j + \underset{q}{\mathrm{Min}}\ \underset{p}{\mathrm{Max}}\ f'(x) \sum_{i,j} a_{ij} p_i q_j \tag{4.3}$$

whence, since $f'(x) \neq 0$,

$$\begin{aligned} 0 &\cong \underset{q}{\mathrm{Min}}\ \underset{p}{\mathrm{Max}} \sum_{i,j} a_{ij} p_i q_j \\ &\cong \underset{p}{\mathrm{Max}}\ \underset{q}{\mathrm{Min}} \sum_{i,j} a_{ij} p_i q_j \end{aligned} \tag{4.4}$$

§5. Nonzero-sum Games

Let us consider a two-person, multi-stage, nonzero-sum game where the first and second players have, respectively, the game matrices

$$A = (a_{ij}), \quad B = (b_{ij}) \tag{5.1}$$

and initially the amounts x and y, respectively.

Let $f(x,y)$ be some criterion function, such as probability of survival, assumed to satisfy the conditions

$$f_x > 0, \quad f_y < 0 \tag{5.2}$$

and assume that

$$\sum_{i,j} a_{ij} p_i q_j, \quad \sum_{i,j} b_{ij} p_i q_j < 0 \tag{5.3}$$

a game of attrition.

Then $f(x,y)$ satisfies the functional equation

$$\begin{aligned} f(x,y) &= \operatorname*{Max}_{p} \operatorname*{Min}_{q} \left[\sum_{i,j} p_i q_j \, f(x+a_{ij}, y+b_{ij}) \right], \quad x,y > 0 \\ &= \operatorname*{Min}_{q} \operatorname*{Max}_{p} \left[\quad \cdots \quad \right] \\ &= 1, \quad x > 0, \quad y < 0 \\ &= 0, \quad x < 0, \quad y > 0 \\ &= 1/2, \quad x = y = 0 \text{ (for the sake of completeness)} \end{aligned} \tag{5.4}$$

Assume as above that x and y are large compared to a_{ij} and b_{ij}. Then

$$f(x+a_{ij}, y+b_{ij}) \cong f(x,y) + a_{ij} f_x + b_{ij} f_y \tag{5.5}$$

Equation (5.4) then yields

$$0 \cong \underset{p}{\mathrm{Max}}\ \underset{q}{\mathrm{Min}} \left[f_x \sum_{i,j} a_{ij}p_iq_j + f_y \sum b_{ij}p_iq_j \right] \tag{5.6}$$

$$\cong \underset{q}{\mathrm{Min}}\ \underset{p}{\mathrm{Max}} \left[\; . \; . \; . \; \right]$$

or

$$\frac{f_x}{f_y} \cong \underset{p}{\mathrm{Min}}\ \underset{q}{\mathrm{Max}} \left[\frac{\sum a_{ij}p_iq_j}{\sum b_{ij}p_iq_j} \right] \tag{5.7}$$

$$\cong \underset{p}{\mathrm{Max}}\ \underset{q}{\mathrm{Min}} \left[\frac{\sum a_{ij}p_iq_j}{\sum b_{ij}p_iq_j} \right]$$

This shows that the single-stage play is approximately governed by the criterion function

$$K(p,q) = \frac{\sum a_{ij}p_iq_j}{\sum b_{ij}p_iq_j} \tag{5.8}$$

That min-max = max-min in (5.7) is a result due to von Neumann. An elegant short proof based on the usual min-max theorem will be found in [4].

We thus have a rationale for the play of large classes of two-person nonzero-sum games.

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